Advanced Microeconomics II
First Tutorial

Nicolas K. Scholtes

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Exercise 15.B.1

Consider an Edgeworth Box economy in which two consumers have locally nonsatiated preferences. Let $x_{\ell i}(p)$ be consumer i's demand for good ℓ at prices $p = (p_1, p_2)$.

(a) Show that $p_1\left(\sum_i x_{1i}(p) - \bar{\omega}_1\right) + p_2\left(\sum_i x_{2i}(p) - \bar{\omega}_2\right) = 0 \quad \forall p$

The competitive budget set $B_i(p) = \{x \in \mathbb{R}^2_+ | p \cdot x \leq p \cdot \omega_i\}$ in vector form for consumers $i = 1, 2$ is given by:

$$\begin{aligned}
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} x_{1i} & x_{2i} \end{pmatrix} \quad &\leq \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} \omega_{1i} & \omega_{2i} \end{pmatrix} \\
p_1\left(x_{1i} - \omega_{1i}\right) \quad &+ \quad p_2\left(x_{2i} - \omega_{2i}\right) \leq 0
\end{aligned}$$

Recall that the (Walrasian) demand function $x_i(p, p \cdot \omega_i)$ must satisfy Walras's Law. That is,

$$p_1\left(x_{1i} - \omega_{1i}\right) + p_2\left(x_{2i} - \omega_{2i}\right) = 0$$

Proof. [1] : Suppose not, $\exists x_i \in B_i(p) | p \cdot x_i < p \cdot \omega i$ $(\neg B)$. Then, by LNS of $\succeq$ $\exists x'_i \in B_i(p)$ with $p \cdot x'_i < p \cdot \omega$ such that $x'_i \succ_i x_i$. However, this contradicts the optimality of x_i $(\neg A)$. Thus, $x_i(p)$ cannot be the demand at $\mathbf{p}$ when $p \cdot x_i < p \cdot \omega i$. □

Having established that $p_1\left(x_{1i} - \omega_{1i}\right) + p_2\left(x_{2i} - \omega_{2i}\right) = 0 \quad \forall i$, we now sum over i:

$$\begin{aligned}
\sum_i p_1\left(x_{1i} - \omega_{1i}\right) \quad &+ \quad p_2\left(x_{2i} - \omega_{2i}\right) = 0 \\
p_1\left[\left(\sum_i x_{1i}\right) - \omega_1\right] \quad &+ \quad p_2\left[\left(\sum_i x_{2i}\right) - \omega_2\right] = 0
\end{aligned}$$

[1]The logical statement, A ⇒ B we are proving is Walrasian demand ⇒ $p \cdot x_i = p \cdot \omega_i$. This is equivalent to proving that 'not A' implies 'not B' i.e. $\neg B \Rightarrow \neg A$

(b) Argue that if the market for good 1 clears at prices $p^\star \gg 0$, then so does the market for good 2; hence $p^\star$ is a Walrasian equilibrium price vector.

Beginning with market-clearing for good **1**. This implies

$$\begin{aligned} x_{11} + x_{12} &= \omega_{11} + \omega_{12} \\ \left(\sum_i x_{1i}\right) &- \omega_1 = 0 \end{aligned}$$

Substituting into our result from (a):

$$\begin{aligned} p_1 \left[\left(\sum_i x_{1i}\right) - \omega_1\right] &+ p_2 \left[\left(\sum_i x_{2i}\right) - \omega_2\right] = 0 \\ p_2 \left[\left(\sum_i x_{2i}\right) - \omega_2\right] &= 0 \end{aligned}$$

And since $p_2 > 0$, we get $\left[\left(\sum_i x_{2i}\right) - \omega_2\right] = 0$ which is the market-clearing condition for good **2**.

Exercise 15.B.2

Consider and Edgeworth box economy in which the consumers have the Cobb-Douglas utility functions $u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha}$ and $u_2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta}$. Consumer i's endowments are $(\omega_{1i}, \omega_{2i}) \gg 0$. Solve for the equilibrium price ratio and allocation. How do these change with a differential change in ω_{11}?

(I) Compute offer curves for each consumer, $OC_i(p) = (x_{1i}^\star, x_{2i}^\star)$

<u>Agent 1 UMP</u>:

$$\begin{aligned} \max_{\{x_{11}, x_{21}\}} \quad u_1(x_{11}, x_{21}) &= x_{11}^{\alpha} x_{21}^{1-\alpha} \\ \text{s.t} \quad &\left\{ \; p_1 x_{11} + p_2 x_{21} \leq p_1 \omega_{11} + p_2 \omega_{21} \right. \end{aligned}$$

For simplicity, let $R_1 = p_1 \omega_{11} + p_2 \omega_{21}$. Recall that when preferences are convex, the optimal consumption can computed by equating the marginal rate of substitution with the price ratio:[2]

$$\frac{\partial u_1 / \partial x_{11}}{\partial u_1 / \partial x_{21}} = \frac{p_1}{p_2} \Rightarrow \frac{\alpha}{1-\alpha} \frac{x_{21}}{x_{11}} = \frac{p_1}{p_2}$$

[2] Tangency between the budget line and the indifference curve is a *necessary* and *sufficient* condition for optimality under convexity of preferences

Substituting the budget constraint allows us to isolate x_{11} as a function of the exogenous parameters

$$\begin{aligned} x_{11} &= \frac{p_2}{p_1}\frac{\alpha}{1-\alpha}\left(\frac{R_1}{p_2} - x_{11}\right) \\ \left(1+\frac{\alpha}{1-\alpha}\right)x_{11} &= \frac{\alpha R_1}{(1-\alpha)p_1} \end{aligned}$$

Simplifying terms, we arrive at the Walrasian demand of consumer 1 for good 1, $x_{11}^{\star}$. Substituting this result into the budget constraint gives us $x_{21}^{\star}$. The two demand functions constitute 1's *Offer curve*:

$$\boxed{OC_1(p) = (x_{11}^{\star}, x_{21}^{\star}) = \left(\frac{\alpha R_1}{p_1}, \frac{(1-\alpha)R_1}{p_2}\right)}$$

The offer curve of agent 2 follows naturally given the symmetry of the utility functions[3]

$$\boxed{OC_2(p) = (x_{12}^{\star}, x_{22}^{\star}) = \left(\frac{\beta R_2}{p_1}, \frac{(1-\beta)R_2}{p_2}\right)}$$

(II) Apply the market clearing condition for good **1**:[4]

$$\begin{aligned} x_{11}^{\star} &+ x_{22}^{\star} = \omega_{11} + \omega_{12} \\ \frac{\alpha R_1}{p_1} &+ \frac{\beta R_2}{p_1} = \omega_{11} + \omega_{12} \\ \frac{\alpha}{p_1}\left[p_1\omega_{11} + p_2\omega_{21}\right] &+ \frac{\beta}{p_1}\left[p_1\omega_{12} + p_2\omega_{22}\right] = \omega_{11} + \omega_{12} \\ \frac{p_2^{\star}}{p_1^{\star}}\left(\alpha\omega_{21} + \beta\omega_{22}\right) &= (1-\alpha)\omega_{11} + (1-\beta)\omega_{12} \end{aligned}$$

Thus, our equilibrium price vector is

$$\boxed{\frac{p_1^{\star}}{p_2^{\star}} = \frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}}$$

This price vector should induce market clearing for good 2 (i.e. we can use this as a check that our calculations were done correctly)

$$\begin{aligned} x_{21}(p_1^{\star}, p_2^{\star}) + x_{22}(p_1^{\star}, p_2^{\star}) &= \frac{(1-\alpha)R_1}{p_2^{\star}} + \frac{(1-\beta)R_2}{p_2^{\star}} \\ &= \frac{\mathbf{p_1^{\star}}}{\mathbf{p_2^{\star}}}\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right] + (1-\alpha)\omega_{21} + (1-\beta)\omega_{22} \\ &= \frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right] + (1-\alpha)\omega_{21} + (1-\beta)\omega_{22} \\ &= \alpha\omega_{21} + \beta\omega_{22} + (1-\alpha)\omega_{21} + (1-\beta)\omega_{22} \\ &= \omega_{21} + \omega_{22} \end{aligned}$$

[3] Note that for agent 2, we have $R_2 = p_1\omega_{12} + p_2\omega_{22}$

[4] Recall: this necessarily implies market clearing for good 2

Having established the equilibrium price vector, substituting this back into the offer curves gives us each consumer's demand of each good at equilibrium

$$
\begin{aligned}
x_{11}(p^*) &= \alpha\omega_{11} + \alpha\frac{\mathbf{p_2^\star}}{\mathbf{p_1^\star}}\omega_{21} \\
&= \alpha\omega_{11} + \alpha\left(\frac{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}{\alpha\omega_{21} + \beta\omega_{22}}\right)\omega_{21} \\
&= \frac{\alpha\left[\alpha\omega_{21} + \beta\omega_{22}\right] + \alpha\omega_{21}\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right]}{\alpha\omega_{21} + \beta\omega_{22}} \\
&= \frac{\alpha(\omega_{11}\omega_{21} + \omega_{21}\omega_{12}) + \alpha\beta(\omega_{11}\omega_{22} - \omega_{21}\omega_{12})}{\alpha\omega_{21} + \beta\omega_{22}} \\
x_{21}(p^*) &= (1-\alpha)\frac{\mathbf{p_1^*}}{\mathbf{p_2^*}}\omega_{11} + (1-\alpha)\omega_{21} \\
&= (1-\alpha)w_{11}\left(\frac{\alpha\omega_{21} + \beta\omega_{22}}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}\right) + (1-\alpha)\omega_{21} \\
&= \frac{(1-\alpha)\omega_{11}\left[\alpha\omega_{21} + \beta\omega_{22}\right] + (1-\alpha)\omega_{21}\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right]}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}} \\
&= \frac{(1-\alpha)\left(\omega_{11}\omega_{21} + \beta\omega_{11}\omega_{22}\right) + (1-\alpha)(1-\beta)\left(\omega_{21}\omega_{12}\right)}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}
\end{aligned}
$$

By symmetry of the utility functions, agent 2's consumption at equilibrium is as follows

$$
\begin{aligned}
x_{12}(p^*) &= \frac{\beta(\omega_{12}\omega_{22} + \omega_{11}\omega_{22}) + \alpha\beta(\omega_{12}\omega_{21} - \omega_{11}\omega_{22})}{\alpha\omega_{21} + \beta\omega_{22}} \\
x_{22}(p^*) &= \frac{(1-\beta)\left(\omega_{12}\omega_{22} + \alpha\omega_{12}\omega_{21}\right) + (1-\alpha)(1-\beta)\left(\omega_{11}\omega_{22}\right)}{(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}}
\end{aligned}
$$

Taking the first derivative w.r.t ω_{11} of the equilibrium values for price and consumption allows us to ascertain the effect of a differential change in ω_{11}

$$
\begin{aligned}
\frac{\partial p^*}{\partial \omega_{11}} &= \frac{(\alpha-1)\left(\alpha\omega_{21} + \beta\omega_{22}\right)}{\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right]^2} < 0 \text{ since } \alpha \in (0,1) \\
\frac{\partial x_{11}(p^*)}{\partial \omega_{11}} &= \frac{\alpha\omega_{21} + \alpha\beta\omega_{22}}{\alpha\omega_{21} + \beta\omega_{22}} > 0 \\
\frac{\partial x_{21}(p^*)}{\partial \omega_{11}} &= \frac{(1-\alpha)(1-\beta)\omega_{12}\left(\alpha\omega_{21} + \beta\omega_{22}\right)}{\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right]^2} > 0 \\
\frac{\partial x_{12}(p^*)}{\partial \omega_{11}} &= \frac{(1-\alpha)\beta\omega_{22}}{\alpha\omega_{21} + \beta\omega_{22}} > 0 \\
\frac{\partial x_{22}(p^*)}{\partial \omega_{11}} &= -\frac{(1-\alpha)(1-\beta)\omega_{12}\left(\alpha\omega_{12} + \beta\omega_{22}\right)}{\left[(1-\alpha)\omega_{11} + (1-\beta)\omega_{12}\right]^2} < 0
\end{aligned}
$$

As consumer 1's endowment of good 1 increases, we see that the relative price of p_1 decreases at equilibrium. Consumer 1's demand for both goods increases as a result of his increases wealth. Consumer 2's demand for good 1 increases as this good becomes more abundant in the economy and his demand for the other good decreases.

Exercise 15.B.6

Compute the equilibria of the following Edgeworth box economy (there is more than one):

$$\begin{aligned}\mathbf{u_1(x_{11}, x_{21})} &= \left(\mathbf{x_{11}^{-2}} + \left(\frac{\mathbf{12}}{\mathbf{37}}\right)^{\mathbf{3}} \mathbf{x_{21}^{-2}}\right)^{-\frac{1}{2}}, \qquad \omega_{\mathbf{1}} = (\mathbf{1}, \mathbf{0}) \\ \mathbf{u_2(x_{12}, x_{22})} &= \left(\left(\frac{\mathbf{12}}{\mathbf{37}}\right)^{\mathbf{3}} \mathbf{x_{12}^{-2}} + \mathbf{x_{22}^{-2}}\right)^{-\frac{1}{2}}, \qquad \omega_{\mathbf{2}} = (\mathbf{0}, \mathbf{1})\end{aligned}$$

Agent 1 UMP:

$$\begin{aligned}\max_{\{x_{11}, x_{21}\}} \quad u(x_{11}, x_{21}) &= \left(x_{11}^{-2} + \left(\frac{12}{37}\right)^3 x_{21}^{-2}\right)^{-\frac{1}{2}} \\ \text{s.t} \quad &\left\{ p_1 x_{11} + p_2 x_{21} \leq p_1 \omega_{11} + p_2 \omega_{21} = p_1 \right.\end{aligned}$$

Setting up the Lagrangean

$$\mathcal{L}(x_{11}, x_{21}, \lambda) = \left(x_{11}^{-2} + \left(\frac{12}{37}\right)^3 x_{21}^{-2}\right)^{-\frac{1}{2}} - \lambda(p_1 x_{11} + p_2 x_{21} - p_1)$$

To simplify, let $A = \dfrac{12}{37}$ .We then derive the following FOCs:

$$\begin{aligned}\partial x_{11} : -\frac{1}{2}\left(x_{11}^{-2} + A^3 x_{21}^{-2}\right)^{-\frac{3}{2}} (-2x_{11}^{-3}) - \lambda p_1 = 0 \quad &\Rightarrow \quad \lambda = \frac{1}{p_1} \cdot = \left(x_{11}^{-2} + A^3 x_{21}^{-2}\right)^{-\frac{3}{2}} x_{11}^{-3} \qquad \textbf{(I)} \\ \partial x_{21} : -\frac{1}{2}\left(x_{11}^{-2} + A^3 x_{21}^{-2}\right)^{-\frac{3}{2}} (-2A^3 x_{21}^{-3}) - \lambda p_2 = 0 \quad &\Rightarrow \quad \lambda = \frac{1}{p_2} \cdot \left(x_{11}^{-2} + A^3 x_{21}^{-2}\right)^{-\frac{3}{2}} A^3 x_{21}^{-3} \qquad \textbf{(II)} \\ \partial \lambda : p_1 x_{11} + p_2 x_{21} = p_1 \quad &\Rightarrow \quad x_{21} = \frac{p_1}{p_2}(1 - x_{11}) \qquad \textbf{(III)}\end{aligned}$$

Setting (I) = (II) then substituting (III) into the resulting expression

$$\begin{aligned}\frac{1}{p_1} x_{11}^{-3} &= \frac{A^3}{p_2} x_{21}^{-3} \\ A\left(\frac{p_1}{p_2}\right)^{\frac{1}{3}} x_{11} &= x_{21} = \frac{p_1}{p_2} - \frac{p_1}{p_2} x_{11}\end{aligned}$$

Which, after some computations, gives us agent 1's offer curve:

$$\boxed{OC_1 = (x_{11}^{\star}, x_{21}^{\star}) = \left(\frac{p_1}{A p_1^{\frac{1}{3}} p_2^{\frac{2}{3}} + p_1}, \frac{A p_1}{A p_2 + p_1^{\frac{2}{3}} p_2^{\frac{2}{3}}}\right)}$$

Using the same reasoning as above for agent 2

$$OC_2 = (x_{12}^{\star}, x_{22}^{\star}) = \left(\frac{Ap_2}{Ap_1 + p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}, \frac{p_2}{Ap_1^{\frac{2}{3}} p_2^{\frac{1}{3}} + p_2} \right)$$

Applying the good 1 market clearing condition to obtain the equilibrium price ratio

$$\begin{aligned} x_{11} + x_{12} = w_{11} + w_{12} &= 1 \\ \frac{p_1}{Ap_1^{\frac{1}{3}} p_2^{\frac{2}{3}} + p_1} + \frac{Ap_2}{Ap_1 + p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}} &= 1 \end{aligned}$$

To simplify, fix $p_2 = 1$ and let $x = p_1^{\frac{1}{3}}$. Then

$$\begin{aligned} \frac{x^2}{A + x^2} + \frac{A}{Ax^3 + x} &= \frac{x^2(Ax^3 + x) + A(A + x^2)}{(A + x^2)(Ax^3 + x)} = 1 \\ Ax^5 + x^3 + A^2 + Ax^2 &= A^2x^3 + Ax + Ax^5 + x^3 \\ A^2x^3 - Ax^2 + Ax - A^2 &= 0 \\ 12x^3 - 37x^2 + 37x - 12 &= 0 \end{aligned}$$

The above cubic equation can be factorized as follows

$$(x - 1)(4x - 3)(3x - 4) = 0$$

Which has 3 roots, $x = \left\{1, \frac{3}{4}, \frac{4}{3}\right\}$. Since $p_1 = x^3$, we get three equilibrium price vectors

$$\frac{p_1^{\star}}{p_2^{\star}} = \left\{1, \left(\frac{3}{4}\right)^3, \left(\frac{4}{3}\right)^3\right\}$$

Exercise 15.C.2

Consider the one-consumer, one-producer economy discussed in Section 15.C. Compute the equilibrium prices, profits, consumptions when the production function is $f(z) = z^{\frac{1}{2}}$, the utility function is $u(x_1, x_2) = \ln(x_1) + \ln(x_2)$, and the endowment of labour is $\bar{L} = 1$.

There are several ways to go about solving this problem. Each involves the same steps however, the order is irrelevant. These are:

1. Consumer optimization: Choose consumption of leisure, x_1 and the commodity, x_2 that maximizes utility subject to the budget constraint.

2. Firm optimization: Choose labour use, z that maximizes profits.

3. Market-clearing condition either in the commodity or labour market.

(I) 2 → 1 → 3

Firm:

$$\max_{z\geq 0} \pi(p,w) = pz^{\frac{1}{2}} - wz$$

Giving us the following FOC

$$\frac{\partial \pi}{\partial z} = \frac{1}{2}pz^{-12} - w = 0 \Rightarrow z = \frac{p^2}{4w^2}$$

Which then gives us the value function

$$\pi(p,w) = pf(\frac{p^2}{4w^2}) - w\frac{p^2}{4w^2} \Rightarrow \pi(p,w) = \frac{p^2}{4w}$$

Consumer[5]:

$$\begin{aligned} \max_{\{x_1,x_2\}} \quad u(x_1,x_2) &= \ln(x_1) + \ln(x_2) \\ \text{s.t} \quad &\left\{ \; px_2 \leq w(1-x_1) + \pi(p,w) \right. \end{aligned}$$

The Lagrangean is

$$\mathcal{L}(x_1,x_2,\lambda) = \ln(x_1) + \ln(x_2) - \lambda(px_2 - w(1-x_1) - \pi(p,w))$$

From which we derive the following FOCs

$$\begin{aligned} \partial x_1 : \frac{1}{x_1} - \lambda w &= 0 \\ \partial x_2 : \frac{1}{x_2} - \lambda p &= 0 \\ \partial \lambda : px_2 = w(1-x_1) + \pi(p,w) &= w - wx_1 + \frac{p^2}{4w} \end{aligned}$$

Which after some computations, gives us the consumers offer curve

$$OC_1(p,w) = (x_1,x_2) = \left(\frac{4w^2+p^2}{8w^2}, \frac{4w^2+p^2}{8wp} \right)$$

Market clearing - Good 2

$$\begin{aligned} x_2 &= f(z) \\ \frac{4w^2+p^2}{8wp} &= \frac{p}{2w} \Rightarrow 4w^2 = 3p^2 \end{aligned}$$

[5] N.B. The consumer consumes x_2 units of the commodity produced by the firm and x_1 units of leisure. As a result, he supplies $\bar{L} - x_1$ of labour hours to the firm, which uses labour as the sole input in the production of x_2. Moreover, we assume that the consumer is the owner of the firm and thus, receives the firm's entire profit. This, along with the wage he is paid for supplying $\bar{L} - x_1$ units of labour, comprises his *wealth* which he can use to purchase x_2 at market price, p

As before, market clearing in the goods market necessarily implies market clearing in the labour market. We can show this by using the above price-wage ratio, $\frac{w}{p} = \sqrt{\frac{3}{4}}$ the consumers leisure choice, x_1 and the producers optimal production,z to prove that $\mathbf{z = 1 - x_1}$

$$RHS = 1 - x_1 = \frac{4w^2 - p^2}{8w^2} = \frac{3p^2 - p^2}{8w^2} = \frac{p^2}{4w^2} = z = LHS$$

Having established the equilibrium price ratio, we can substitute into our expressions for z, x_1, x_2 to obtain

$$\boxed{(z, \pi, x_1, x_2) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{\sqrt{3}}\right)}$$

Given that w and p are not independent (recall: $4w^2 = 3p^2$), we need to fix one in order to determine the other and compute the profits at equilibrium. Letting $p^* = 1$, we get the following values for w^* and $\pi(1, w^*)$:

$$\boxed{(p^*, w^*, \pi(p^*, w^*)) = \left(1, \frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{3}}\right)}$$

(II) $\mathbf{3 \to 1 \to 2}$

Another way to go about solving the problem is to substitute the market clearing conditions $(x_2^* = f(z^*), x_1^* = 1 - z^*)$ directly into one of the agents' maximisation programmes:

$$\max_{z \geq 0} u(x_1, x_2) = u(1 - z, z^{\frac{1}{2}}) = \ln(1 - z) + \ln(z^{\frac{1}{2}})$$

Differentiating w.r.t z gives us the following optimality conditions

$$\frac{\partial u}{\partial z} = \frac{1}{z - 1} + \frac{1}{2z} = 0 \Rightarrow z^* = \frac{1}{3}$$

Which then gives us

$$\begin{aligned} x_2^* &= f(z^*) = \left(\frac{1}{3}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \\ x_1^* &= 1 - z^* = \frac{2}{3} \end{aligned}$$

Profit maximisation for the firm then allows us to compute the remaining parameters.

Advanced Microeconomics II
Second Tutorial

Nicolas K. Scholtes

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Exercise 16.C.2

Suppose that the preference relation $\succeq_i$ is locally nonsatiated and $x_i^\star$ is maximal for $\succeq_i$ in set $\{x_i \in X_i : p \cdot x_i \leq w_i\}$. Prove that the following property holds: "If $x_i \succeq_i x_i^\star$ then $p \cdot x_i \geq w_i$."

This property is required in order to establish the conditions for Pareto-efficient allocations in the First-Welfare Theorem. Given an economy $(\{X_i, \succeq_i\}_i, \{Y_j\}_j, \bar{\omega})$, a price equilibrium with transfers [1] must satisfy a number of conditions, one of which is that the equilibrium allocation $(x^\star, y^\star)$ must be preference and profit maximizing for consumers and firms respectively. The former implies that if there is an allocation that is *strictly* preferred to the equilibrium allocation for any i, then that allocation should lie *outside* i's budget set. That is,

$$\text{If } \exists x_i \in X_i | x_i \succ_i x_i^\star \Rightarrow p \cdot x_i > w_i$$

Given the above statement for strict preference relations, we are asked to establish a similar property for the weak preference relation. Using a contradiction argument[2] along with the LNS property, the proof is as follows

Proof. Suppose *a contrario* that, $\exists x_i' \in X_i : x_i' \succeq_i x_i^\star$ and $p \cdot x_i < w_i$
By **LNS**: $\exists x_i'' \in X_i$ and $\epsilon > 0$ such that $\|x_i'' - x_i'\| < \epsilon$, $x_i'' \succ_i x_i'$ and $p \cdot x_i'' < w_i$

By **Transitivity of $\succeq_i$**: $x_i'' \succ_i x_i' \succeq_i x_i^\star \Rightarrow x_i'' \succ_i x_i^\star$

Given that $x_i'' \succ_i x_i^\star$ and $p \cdot x_i'' < w_i$, it is apparent that $x_i^\star$ is no longer the maximiser in the budget set.

Thus, the initial optimality of $x_i^\star$ is violated. If $x_i^\star$ is the maximiser, then it follows that $p \cdot x_i \geq w_i$ □

[1] That is, we assume a social planner who can carry out lump sum redistributions of wealth amongst the agents under the condition that $\sum_i w_i = p\bar{\omega} + p \sum_j y_j^\star$

[2] Setting up the question as a logical statement of type A ⇒ B: If $x_i \succeq_i x_i^\star \Rightarrow p \cdot x_i \geq w_i$

Exercise 16.C.3

In this exercise you are asked to establish the first welfare theorem under a set of assumptions compatible with satiation. Suppose that every $\mathbf{X_i}$ is nonempty and convex and that every $\succeq_{\mathbf{i}}$ is strictly convex (i.e. if $\mathbf{x_i'} \succeq_{\mathbf{i}} \mathbf{x_i}$ and $\mathbf{x_i'} \neq \mathbf{x_i}$ then $\alpha\mathbf{x_i'} + (\mathbf{1}-\alpha)\mathbf{x_i} \succ_{\mathbf{i}} \mathbf{x_i}$ whenever $\mathbf{0} < \alpha < \mathbf{1}$). Prove the following:

(a) Every i can have at most one satiation point and preferences are locally nonsatiated at any consumption bundle different from the single global satiation point.

Definition: A satiation point is a quantity of consumption where any further changes result in a decrease in the well-being of the consumer. Moreover, global satiation implies a maximal level of utility in the absence of any budget constraint. Unlike **local** nonsatiation, which implies that within a **neigbourhood** of any allocation, a strictly preferred allocation can be found, global satiation implies that there exists a point (or set of points) for which the consumer does not wish to change his consumption plan, regardless of what alternative bundles are proposed.

We start with the first assertion which, mathematically, states that

$$\exists! y_i \in X_i : y_i \succ_i x_i \forall x_i \in X_i$$

To prove that more than one satiation points is incompatible with the specifications of the consumption set and preferences (X_i nonempty, convex and $\succeq_i$ strictly convex), we construct a proof stating that an economy with 2 GSPs **cannot** exist under the properties of $(X_i, \succeq_i)$ given above.

Proof. Suppose there are 2 GSPs. That is, $\exists x_i, x_i' \in X_i : x_i \sim x_i'$ and $x_i \neq x_i'$. Given the mathematical definition of GSPs given above, we must have that $x_i \succ_i y_i$ and $x_i' \succ_i y_i \ \forall y_i \in X_i$.

By **strict convexity** of preferences: $x_i'' = \alpha x_i + (1-\alpha)x_i' \succ_i x_i \sim x_i' \forall \alpha \in (0,1)$

Using this assumption, we have established that the any allocation on the line segment connecting x_i and x_i' is strictly preferred to the original allocations. This imposes a degree of curvature to the indifference map since the set of all possiblex_i'' (determined by the value of α must lie in the strict upper contour set of the indifference curve linking x_i and x_i'.[3]

Convexity of X_i: Recall that a set, A in Euclidean space is convex if the point $\alpha x + (1-\alpha)x' \in A$ whenever $x, x' \in A$ and $\alpha \in [0,1]$. Given the convexity of the set, $x_i'' = \alpha x_i + (1-\alpha)x_i' \in X_i | x_i, x_i' \in A$

Thus, we have established two facts; $\exists x_i'' \in X_i : x_i'' \succ_i x_i, x_i'$ and $x_i^{''} \in X_i$. As a result, x_i and x_i' cannot be GSPs as we are able to locate an allocation within the consumption set that is strictly preferred by the consumer.

[3] see MWG pp. 44-45 for a discussion of convex preferences

For the second assertion, we show that preferences are locally nonsatiated for any allocation different from the unique GSP. Given that $x_i^\star$ is the GSP, taking any $x_i \neq x_i^\star$ and $x_i \in X_i$, we must have that $x_i^\star \succeq_i x_i$. Moreover, since x_i is not the GSP, $\exists x_i' \in X_i : x_i' \succeq_i x_i$.

Since $x_i, x_i' \in X_i$, strict convexity of $\succeq_i$ tells us that $x_i'' = \alpha x_i + (1-\alpha)x_i' \succ_i x_i$ for any $\alpha \in (0,1)$. Thus, we have the following preference ordering: $x_i'' \succ_i x_i' \succeq_i x_i$ which by transitivity, implies that $x_i'' \succ_i x_i$. Finally, given that x_i'' is defined for a continuum of values, we can state that $\|x_i'' - x_i\| \leq \epsilon$, $\forall \epsilon > 0$. This is precisely the definition of local nonsatiation.

□

(b) Any price equilibrium with transfers is a Pareto optimum (= the first welfare theorem)

Using assertions made in (a), we analyze first the case where $x_i^\star$ is *not* a GSP followed by the case where it is a GSP.

(i) If $x_i^\star$ is not a GSP, we know from the second assertion of (a) that preferences are locally nonsatiated. Furthermore, we proved in Ex. 16.C.2, that under LNS, if $\exists x_i \succeq_i x_i^\star$ then $p \cdot x_i \geq w_i$ where $x_i^\star$ is i's consumption at equilibrium.

We prove this by contradiction. Suppose, *a contrario* that $x^\star$ is not a Pareto optimum. Using the statement above, this implies the following:

$$\exists (x,y) \in X_1 \times \ldots \times X_I \times Y_1 \times \ldots \times Y_J : \begin{cases} x_i \geq x_i^\star & \forall i \Rightarrow p \cdot x_i \geq w_i \\ x_i > x_i^\star & \text{for some } i \Rightarrow p \cdot x_i > w_i \end{cases}$$

Summing over all the i's, the RHS of the above assertion implies that $\sum_i p \cdot x_i > \sum_i w_i$. Recall that the total final wealth in the economy must be equal to the sum of the market value of the endowments and the market value of equilibrium production. This gives rise to the following inequality

$$\sum_i p \cdot x_i > \sum_i w_i = p \cdot \bar{w} + \sum_j p \cdot y_j^\star \geq p \cdot \bar{w} + \sum_j p \cdot y_j$$

Where the last inequality follows from the the fact that the equilibrium production $y_j^\star$ is profit-maximising in Y_j. Taking the two ends of the above chain of inequalities and cancelling the price terms, we get that $\sum_i x_i > \sum_i w_i + \sum_j w_j$. This applies to the Pareto-dominating allocation (x,y). Obviously, this allocation violates the feasibility constraint and thus, cannot be a price equilibrium with transfers.

(ii) If $x_i^\star$ *is* a GSP, we know that this point must be unique by the first assertion of part (a). Moreover, since the GSP corresponds to the maximum level of satisfaction attainable by the consumers, any deviation from this point would render them worse off. Hence, if $x_i^\star$ is a GSP, then there cannot exist an alternative allocation (x,y) that Pareto-dominates it. Thus, $x_i^\star$ is Pareto optimum.

Exercise 16.E.2

Show that the utility possibility set U of an economy with convex production and consumption sets and with concave utility functions is convex.

Utility possibility set: Recall that a utility function maps a vector into the set of positive real numbers. That is, $u : \mathbb{R}^L \to \mathbb{R}_+$. The utility possibility *set*, defined as the utility levels an economy can achieve for any feasible allocation, is constructed by taking the image of this mapping and all vectors less than or equal to the image. As a result, the UPS for any feasible allocation (x,y), is such that $u_i \leq u_i(x_i)$. A graphical interpretation of the UPS for an Edgeworth economy is provided in *fig. 1*

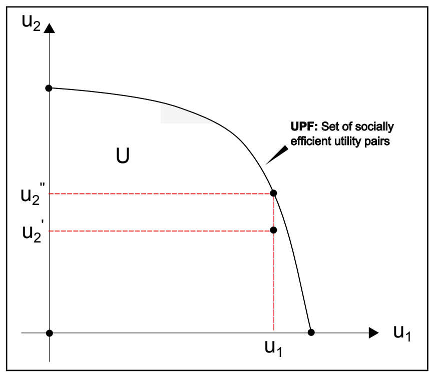

Figure 1: Edgeworth economy in utility space

Where the coordinate defined by $(u_1, u_2^{''})$ lies on the utility possibility frontier and thus, by definition, is Pareto efficient. By contrast, at $(u_1, u_2^{'})$, given agent 1's action agent 2 can improve his utility without decreasing that of agent 1. Thus, it is not Pareto efficient.

To prove that U is a convex set given the information in the problem statement, let us define two allocations , x, $x^{'} \in X$ whose respective utility functions are in the economy's UPS.[4] That is, $u_i \leq u_i(x_i)$ and $u_i^{'} \leq u_i(x_i^{'})$ $\forall i$ with $u, u^{'} \in U$. By convexity of X_i, the convex combination of any two elements of X_i must be in X_i: $x^{''} = \alpha x + (1-\alpha)x^{''}$ and $x^{''} \in X$. $\forall \alpha \in [0,1]$

In order to prove convexity of U, we have to show that for $u^{''} = \alpha u + (1-\alpha)u^{'}$, $u^{''} \in U$ $\forall \alpha \in [0,1]$. This amounts to showing that the condition for inclusion in U is satisfied. That is, $u_i^{''} \leq u_i(x_i^{''})$ To do this, we set up the following system of inequalities

$$u_i^{"} = \lambda u + (1-\lambda)u^{'} \leq \lambda u_i(x_i) + (1-\lambda)u_i(x_i^{'}) \leq u_i(\alpha x_i + (1-\alpha)x_i^{'}) = u_i(x_i^{"})$$

Where the first inequality follows from the fact that $u, u^{'} \in U$ by definition thereby allowing us to use the inequalities defining the UPS. The second follows from concavity of the utility *function* which says that any point on the line segment connecting x and $x^{'}$ will always be less than or equal to the utility associated to the convex combination of those points.

The above system of inequalities simplifies to $u_i^{"} \leq u_i(x_i^{"})$ which implies that $u^{"} \in U$. This concludes the proof.

[4] we ignore the production side of the economy for which the following intuitions are unchanged

Exercise 17.D.1

Consider an exchange economy with two commodities and two consumers. Both consumers have homothetic preferences of the constant elasticity variety. Moreover, the elasticity of substitution is the same for both consumers and is small (i.e., goods are close to perfect complements).

$\mathbf{u_1(x_{11}, x_{21}) = (2x_{11}^{\rho} + x_{21}^{\rho})^{\frac{1}{\rho}}}$ and $\mathbf{u_2(x_{12}, x_{22}) = (x_{12}^{\rho} + 2x_{22}^{\rho})^{\frac{1}{\rho}}}$

and $\rho = -4$**. The endowments are** $\omega_1 = (1, 0)$ $\omega_2 = (0, 1)$**.**
Compute the excess demand function of this economy and verify that there are multiple equilibria.

Recall that preference orderings are invariant under monotonic transformations. Thus, for any strictly increasing function $f(\cdot) : \mathbb{R} \to \mathbb{R}$, $\tilde{u}(x) = f(u(x))$ represents the same preferences. We apply the following transformation:

$$\tilde{u}(x) = \frac{1}{\rho} u(x)^{\rho} \Rightarrow \begin{cases} \tilde{u_1}(x_{11}, x_{21}) = \frac{1}{\rho}(2x_{11}^{\rho} + x_{21}^{\rho}) \\ \tilde{u_2}(x_{12}, x_{22}) = \frac{1}{\rho}(x_{12}^{\rho} + 2x_{22}^{\rho}) \end{cases}$$

(I) Computing Walrasian demand functions of each agent

<u>Agent 1 UMP</u>

$$\max_{\{x_{11}, x_{21}\}} \quad \tilde{u_1}(x_{11}, x_{21}) = \frac{1}{\rho}(2x_{11}^{\rho} + x_{21}^{\rho})$$

$$\text{s.t} \qquad \left\{ \; p_1 x_{11} + p_2 x_{21} \leq p_1 \omega_{11} + p_2 \omega_{21} = p_1 \right.$$

Giving us the following Lagrangean

$$\mathcal{L}(x_{11}, x_{21}, \lambda) = \frac{1}{\rho}(2x_{11}^{\rho} + x_{21}^{\rho}) - \lambda(p_1 x_{11} + p_2 x_{21} - p_1)$$

From which we derive the following FOCs

$$\begin{aligned} \partial x_{11} : 2x_{11}^{\rho-1} - \lambda p_1 = 0 \quad &\Rightarrow \quad \lambda = \frac{2}{p_1} x_{11}^{\rho-1} (\mathbf{I}) \\ \partial x_{21} : x_{21}^{\rho-1} - \lambda p_2 = 0 \quad &\Rightarrow \quad \lambda = \frac{1}{p_2} x_{21}^{\rho-1} \; (\mathbf{II}) \\ \partial \lambda : p_1 x_{11} + p_2 x_{21} = p_1 \quad &\Rightarrow \quad x_{21} = \frac{p_1}{p_2}(1 - x_{11}) \qquad (\mathbf{III}) \end{aligned}$$

Setting (**I**) = (**II**) then substituting in (**III**) allows us to compute $x_{11}^{\star}(p_1, p_2)$

$$x_{11} = \left(\frac{p_1}{2p_2}\right)^{\frac{1}{\rho-1}} \cdot x_{21} = \left(\frac{p_1}{2p_2}\right)^{\frac{1}{\rho-1}} \cdot \frac{p_1}{p_2}(1 - x_{11}) = \frac{p_1^{\frac{\rho}{\rho-1}}}{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}}$$

$$x_{11} \cdot \left(1 + \frac{p_1^{\frac{\rho}{\rho-1}}}{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}}\right) = x_{11} \cdot \left(\frac{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} + p_1^{\frac{\rho}{\rho-1}}}{\cancel{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}}}\right) = \frac{p_1^{\frac{\rho}{\rho-1}}}{\cancel{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}}}$$

$$x_{11} = \frac{p_1^{\frac{\rho}{\rho-1}}}{2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} + p_1^{\frac{\rho}{\rho-1}}} = \frac{p_1^{\frac{1}{\rho-1}} p_1}{p_1^{\frac{1}{\rho-1}} p_1 + 2^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}}} = \frac{\cancel{p_1^{\frac{1}{\rho-1}}} p_1}{\cancel{p_1^{\frac{1}{\rho-1}}} (p_1 + 2^{\frac{1}{\rho-1}} p_1^{\frac{1}{1-\rho}} p_2^{\frac{1}{\rho-1}} p_2)}$$

Grouping like-terms, we arrive at the following demand function

$$\boxed{x_{11} = \frac{p_1}{p_1 + \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}} p_2}}$$

Using the budget constraint in **(III)**, we compute $x_{21}^{\star}(p_1, p_2)$

$$x_{21} = \frac{p_1}{p_2}(1 - x_{11}) = \frac{p_1}{p_2}\left(1 - \frac{p_1}{p_1 + \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}} p_2}\right) = \frac{p_1 p_1^{\frac{1}{1-\rho}} \left(\frac{1}{2}\right)^{\frac{1}{1-\rho}} p_2^1 \rho - 1}{p_1 + \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}} p_2}$$

$$\boxed{x_{21} = \frac{p_1 \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}}}{p_1 + \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}} p_2}}$$

Following the same procedure for agent 2, we compute his demand for goods 1 and 2

$$\boxed{(x_{12}, x_{22}) = \left(\frac{p_2 \left(\frac{p_2}{2p_1}\right)^{\frac{1}{1-\rho}}}{p_2 + \left(\frac{p_2}{2p_1}\right)^{\frac{1}{1-\rho}} p_1}, \frac{p_2}{p_2 + \left(\frac{p_2}{2p_1}\right)^{\frac{1}{1-\rho}} p_1}\right)}$$

(II) Aggregate excess demand for good 1 = 0 (= Market clearing)

$$\begin{aligned} z_1(p_1, p_2) = z_{11} + z_{12} &= x_{11} - \omega_{11} + x_{12} - \omega_{12} = 0 \\ &= \frac{p_1}{p_1 + \left(\frac{p_1}{2p_2}\right)^{\frac{1}{1-\rho}} p_2} - 1 + \frac{p_2 \left(\frac{p_2}{2p_1}\right)^{\frac{1}{1-\rho}}}{p_2 + \left(\frac{p_2}{2p_1}\right)^{\frac{1}{1-\rho}} p_1} = 0 \end{aligned}$$

Since $z(.,.)$ is homogenous of degree 0, we can treat good 2 as the numéraire leaving p_1 as the only independent variable; $z(p_1, p_2) = z(p_1/p_2, 1) = z(p_1, 1)$. Using this formulation allows us to simplify the above to

$$z_1(p_1,1) = \frac{p_1}{p_1 + \left(\frac{p_1}{2}\right)^{\frac{1}{1-\rho}}} + \frac{p_2\left(\dfrac{1}{2p_1}\right)^{\frac{1}{1-\rho}}}{1+\left(\frac{1}{2p_1}\right)^{\frac{1}{1-\rho}} p_1} = \frac{p_1}{p_1 + \left(\frac{p_1}{2}\right)^{\frac{1}{1-\rho}}} + \frac{1}{p_1 + (2p_1)^{\frac{1}{1-\rho}}} = 1$$

$$\frac{p_1\left[\left(p_1 + (2p_1)^{\frac{1}{1-\rho}}\right)\right] + p_1 + \left(\frac{p_1}{2}\right)^{\frac{1}{1-\rho}}}{\left(p_1 + \left(\frac{p_1}{2}\right)^{\frac{1}{1-\rho}}\right)\left(p_1 + (2p_1)^{\frac{1}{1-\rho}}\right)} = 1$$

Multiplying out the terms and substituting in $\rho = -4$, we get

$$\begin{aligned}
\cancel{p_1^2} + \cancel{p_1(2p_1)^{1/5}} + p_1 + \left(\frac{1}{2}\right)^{1/5} p_1^{1/5} &= \cancel{p_1^2} + \cancel{p_1(2p_1)^{1/5}} + p_1\left(\frac{p_1}{2}\right)^{1/5} + (2p_1)^{1/5}\left(\frac{p_1}{2}\right)^{1/5} \\
p_1 + \left(\frac{1}{2}\right)^{1/5} p_1^{1/5} &= \left(\frac{1}{2}\right)^{1/5} p_1^{6/5} + p_1^{2/5} \\
\text{Setting} \quad t = p_1^{1/5} : t^5 + \left(\frac{1}{2}\right)^{1/5} t &= \left(\frac{1}{2}\right)^{1/5} t^6 + a
\end{aligned}$$

Removal of the lowest common multiplier gives us the following polynomial of order 5:

$$\boxed{t^5 - 2^{1/5}t^4 + 2^{1/5}t - 1 = 0}$$

The first root (corresponding to one of the possible equilibrium price vectors) can be found by setting $t = 1$. That is, $f(1) = (1)^5 - 2^{1/5}(1)^4 + 2^{1/5}(1) - 1 = 0$. Thus $t = 1 \Rightarrow p_1/p_2 = 1$ is an equilibrium price vector. The following plot of $z(\cdot, 1)$ vs. p_1 can be used to locate the remaining roots:

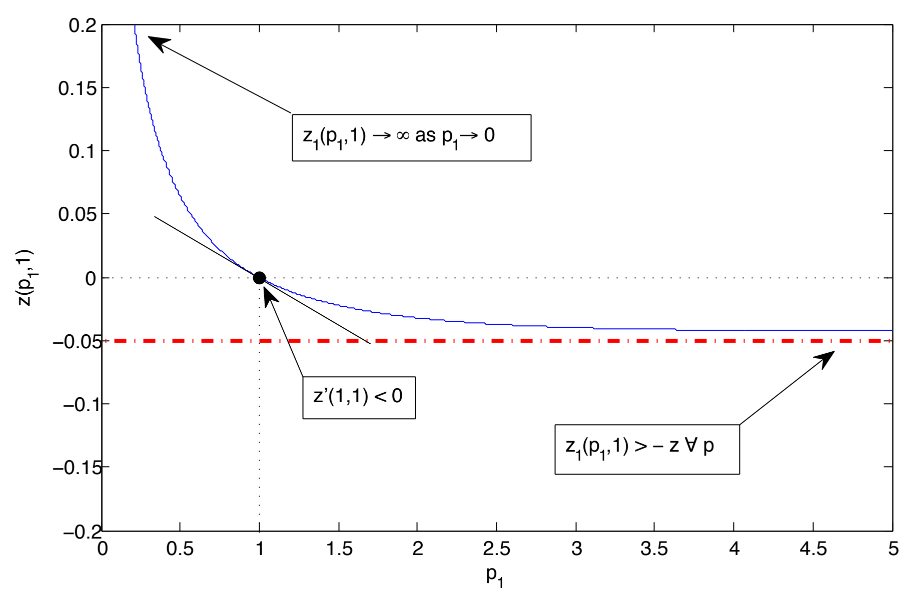

Figure 2: Proof of existence of the unique equilibrium

Evidently, there are no other equilibrium price vectors. Indeed, computation of the roots of the quintic equation in t given above yields 5 solutions, four of which are complex numbers and the remaining one being the root calculated above.

Exercise 19.D.4

Consider a three period economy t = 0, 1, 2, in which at t = 0 the economy splits into two branches and at t = 1 every branch splits again into two. There are *H* physical commodities and consumption can take place at the three dates

(a) Describe the Arrow Debreu equilibrium problem for this economy

In this problem, information is released gradually over time with agents learning more regarding the possible final state as time unfolds. The information tree is captured in *fig 3.* In order to be consistent with the timeless approach of the basic AD setup, it is necessary to restrict the set of feasible consumption plans based on the information structure.

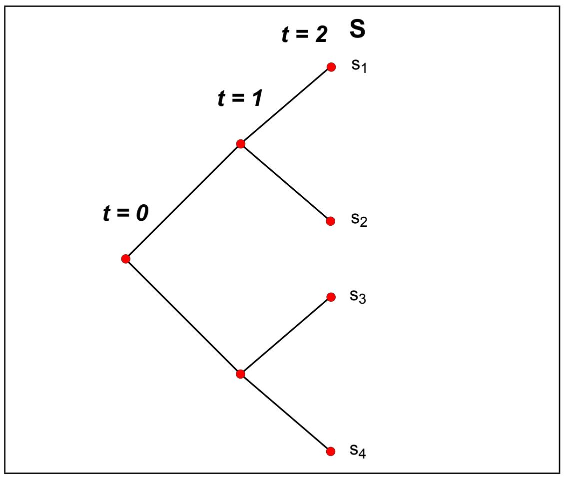

Figure 3: Information tree

At each node of the tree, consumption decisions are determined by the information (regarding the possible realizations of the final state) up to that point. Moreover, information acquired at a particular node is retained for all successor nodes. This allows us to define the *information* structure at each t:

$$\begin{aligned}
\varphi_0 &= (\{s_1, s_2, s_3, s_4\}) \\
\varphi_1 &= (\{s_1, s_2\}, \{s_3, s_4\}) \\
\varphi_2 &= (\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\})
\end{aligned}$$

At $t = 0$, no information is available so agents are unable to distinguish between the states. At $t = 1$, new information allows agents to determine which node (of the two possible nodes) they are located at. Furthermore, it allows them to eliminate certain states from the possible realizations. For example, an agent finding himself at the top node at $t = 1$ knows that s_3 and s_4 will not occur however, he is still unable to distinguish between s_1 and s_2.

In order to be consistent with the AD framework, agents should not be able to condition their consumption on a state they are not sure will be realized. This is known as *(Lebesgue)-measurability.* Formally, we say that a vector $z \in \mathbb{R}^{H(T+1)S}$ of consumption plans is measurable with respect to the family of information partitions $(\varphi_0, ..., \varphi_T)$ if, for any lts and lts', $z_{lts} = z_{lts'}$ whenever s and s' belong to the same element of φ_t. As a result, agents located at the top node at $t = 1$ could not fix different consumption plans contingent on s_1 or s_2 being realized as he does not have the information to distinguish between them. As a result, the set of *admissible* date-event pairs, tE (where consumption plans are formed) are as follows

$$\mathcal{A} = \begin{cases} a_0 = (0, \{s_1, s_2, s_3, s_4\} \\ a_1 = (1, \{s_1, s_2\}); (1, \{s_3, s_4\}) \\ a_2 = (2, \{s_1\}); (2, \{s_2\}); (2, \{s_3\}); (2, \{s_4\}) \end{cases}$$

Arrow Debreu Equilibrium: An allocation defined over the set of admissible date events, $tE \in \mathcal{A}$ $(x^\star_{tE1}, ..., x^\star_{tEI})$ with $x_{tEi} \in \mathbb{R}^H_+$ and a price vector $p_{tE} \in \mathbb{R}^H_+$ constitute an AD equilibrium if

(i) For every i and $tE \in \mathcal{A}$, $x_tEi^\star$ is maximal for $\succeq_i$ in the budget set $\{x_{tEi} \in X_{tEi} : \sum_{tE \in \mathcal{A}} p_{tE} \cdot x_{tEi} \leq \sum_{tE \in \mathcal{A}} p_{tE} \cdot \omega_{tEi}\}$ $\forall tE \in \mathcal{A}$. Intuitively, this implies that the sum of the market values of the contingent claims made by i at every date-event tE must be less than the total market value of his wealth (Note that this restriction does not apply to individual date events).

(ii) $\sum_i x^\star_{tEi} = \sum_i \omega^\star_{tEi}$ $\forall tE in \mathcal{A}$. The equilibrium must be such that markets clear (all consumers achieve their desired trades at the going market prices).

(b) Describe the Radner equilibrium problem for this economy. Suppose that at $t = 0$ and $t = 1$ there are contingent markets for the delivery of one unit of the first physical good at the following date

Recall that in the Radner framework, good 1 is traded on contingent markets (in this case, at $t = 0$ for date-event contingent delivery at $t = 1$ and at $t = 1$ for delivery at $t = 2$). Thus, we establish the price vector, q and trading plan z_i for these contingent commodities at date-event tE :

$$\begin{aligned} q_{tE} &= \{q_{tE}(t+1, E')\} \forall tE \in \mathcal{A} \\ z^\star_i &= \{z^\star_{tEi}(t+1, E')\} \forall tE \in \mathcal{A} \end{aligned}$$

Where $E' \in \varphi_{t+1}$ is the set of successor nodes to E. Thus, $\{q_{tE}(t+1, E')\}$ is the contingent price vector of one unit of *good 1* delivered at $t+1$ if event E' (being one of the possible successor nodes at $t+1$) is revealed and $\{z^\star_{tEi}(t+1, E')\}$ is the (date-event) contingent trade deliverable at $t+1$.

In addition to the good 1 contingent markets, the Radner framework allows for spot markets for all H goods at all date-events. Thus, we must define a spot market budget constraint for each date-event, taking into account the set of successor and predecessor nodes

<u>$t = 0$</u>: Spot markets for all goods, contingent markets for good 1 to be delivered at $t = 1$

$$\begin{aligned} &(i) \quad \sum_{tE' \in \{a_1\}} q_{0E}(1, E') \cdot z_{0Ei}(1, E') \leq 0 \\ &(ii) \quad p_{0E} \cdot x_{0Ei} + \sum_{1, E' \in a_1} q_{0E}(1, E') \cdot z_{0Ei}(1, E') \leq p_{0E} \cdot \omega_{0Ei} \end{aligned}$$

Where $E^{'} \in \varphi_1$

<u>$t = 1$</u>: Spot markets at 2 date events, delivery of good 1 from contingent trades at $t = 0$, contingent markets for good 1 to be delivered at $t = 2$

$$\begin{aligned} &(i) \quad \sum_{tE' \in \{a_2\}} q_{1E}(2, E') \cdot z_{1Ei}(2, E') \leq 0 \\ &(ii) \quad p_{1E} \cdot x_{1Ei} + \sum_{tE' \in \{a_2\}} q_{1E}(2, E') \cdot z_{1Ei}(2, E') \leq p_{1E}\omega_{1Ei} + p_{1E} \cdot z_{0,E_0,i} \qquad \forall 1E \in a_1 \end{aligned}$$

Where $E \in \varphi_1$, $E^{'} \in \varphi_2$ (the set of possible successor date events to E) and E_0 is the known predecessor to E that occurred at $t = 0$. Notice that condition *ii* corresponds to two budget constraints that occur at each node at $t = 1$. These two constraints are defined by date-event $1E \in a_1$.

<u>$t = 2$</u>: Spot markets at 4 date events, delivery of good 1 from contingent trades at $t = 1$, no contingent markets for future dates (terminal nodes)

$$\begin{aligned} &(i) \quad p_{2E} \cdot x_{2Ei} \leq p_{2E} \cdot \omega_{2Ei} + p_{1,2E} z_{1,E_1,i}(2, E) \\ &(ii) \quad p_{2E'} \cdot x_{2E'i} \leq p_{2E'} \cdot \omega_{2E'i} + p_{1,2E'} z_{1,E_1',i}(2, E') \end{aligned}$$

Note that at $t = 1$, the agent is able to distinguish his position on the information tree and thus, has improved knowledge regarding the final outcome. For this reason, two separate budget constraints are required depending on the observed realization at $t = 1$. The first constraint corresponds to the case where the date event $1E \in a_1$ was achieved at $t = 1$ corresponding to the top node in *fig. 3*. Thus date-event $tE \in (2, \{1\}, \{2\})$ are the possible realizations when $tE_1 \in (1, \{1, 2\})$ was observed and $tE \in (2, \{3\}, \{4\})$ are the possible realizations when $tE_1^{'} \in (1, \{3, 4\})$ was observed. As required, this corresponds to four budget constraints for each terminal node.

Having defined the Radner budget sets, the rational consumer now chooses his spot and contingent trading vectors in order to maximise his utility under the constraints given above.

(c) Argue that the proposition of 19.D.1 remains valid

Recall that one of the main conditions of the Radner equilbrium is that agents have rational expectations. That is to say, planned (expected) behaviour equals actual behaviour thereby rendering irrelevant, as in the AD framework. Proving that the AD and Radner *prove* that the budget sets are equivalent amounts to proving the following condition: $B_i^{AD} \subseteq B_i^R$ and $B_i^R \subseteq B_i^{AD}$. If both of these statement are true, then necessarily $B_i^{AD} = B_i^R$. As a result, the proof is carried out in two steps.

Following the proof in MWG, we claim that if there exists a one-to-one transformation between the two equilibria such that any price vector in the AD equilibrium can be transformed into a price vector in the Radner equilibrium and vice versa, then the two budget sets are equal. This comprises part (i) of the proof.

Part (ii) of the proof states that the budget sets of the consumers will be the same under the aforementioned transformation. This amounts to proving the following condition: $B_i^{AD} \subseteq B_i^R$ and $B_i^R \subseteq B_i^{AD}$. If both of these statement are true, then necessarily $B_i^{AD} = B_i^R$.

As in MWG, we set the spot price vector, p_1 of the first good[5] at every date-event equal to the contingent first-good price, q at *the same date-event*. Analytically, this states that $q_{t,E}(t+1, E') = p_{1,(t+1,E')}$ where $(t, E) \in \mathcal{A} \setminus \{a_2\}$[6], $(t + 1, E') \in \mathcal{A}$ and $E' \subset E$.

The rest of the proof follows naturally from MWG p. 697, generalized to the gradual release of information scenario (i.e. consumption defined over date-events rather than states).

[5]Recall that all H goods are traded on spot markets

[6]since contingent markets open at $t = 0$ and $t = 1$ only as specified in the problem statement

Advanced Microeconomics II
Third Tutorial

Nicolas K. Scholtes

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Q1. In Rubinstein's alternating offers bargaining game, if agent 1's discount factor, $\delta_1 = \frac{2}{3}$ and agent 2's $\delta_2 = \frac{4}{5}$,

(1) what is the subgame perfect equilibrium agreement they would reach if agent 1 makes the first offer?

Firstly, taking two allocations x_2 and x_1 such that $x_2 < x_1$, consider the following profile of strategies

1. for a:
   - Claim x_1 whenever b's last offer was strictly smaller than x_2
   - Accept any offer $x \geq x_2$
2. for b:
   - Offer x_2 whenever b's last claim was strictly greater than x^1
   - Accept any claim $x \leq x_1$

We claim that the above profile constitutes a SPNE under a set of conditions. That is, we assume that agents do not *regret* accepting or rejecting an offer/claim when it was made. Specifically, we require that a player's strategy (which, recall specifies actions at every period, for *every* possible history up to that, including those off the equilibrium path)[1] is optimal in the game beginning at *every* node of the tree. This gives rise to two preference relations yielding the no regret condition

No regret accepting an offer ⇒ What the agent gets by accepting an offer must be *at least as good as* what he would have got rejecting the offer and waiting one period to have his own offer accepted.

No regret rejecting an offer ⇒ What an agent gets by rejecting an offer (that is, the counteroffer he makes) must be *at least as good as* what he would have got had he accepted the offer in the previous round.

[1]see P.38 Ch.2 in 'Bargaining and Markets' by Osborne & Rubinstein

Putting the two above conditions together implies that an agent is *indifferent* between accepting what is offered or waiting 1 period to get his own offer accepted. Given that agent 1 makes the first move: *Claims* x_1 for himself according to his strategy. At any t, this implies that he must have rejected a previous *offer* from agent 2 at $t-1$, a decision he must not regret:

$$\delta_1^t x_1 = \delta_1^{t-1} x_2 \Rightarrow x_2 = \delta_1 x_1$$

Agent 2, following his strategy accepts A's claim of x_1 at t, leaving him with $1-x_1$. No regret implies that he is indifferent between accepting $1-x_1$ at t and rejecting and having his offer accepted at $t+1$:

$$\delta_2^t(1-x_1) = \delta_2^{t+1}(1-x_2) \Rightarrow 1-x_1 = \delta_2(1-x_2)$$

This gives us a system of 2 equations and 2 unknowns which we can solve for x_1, x_2

$$\begin{aligned} 1-x_1 = \delta_2(1-x_2) &= \delta_2(1-\delta_1 x_1) \\ &= \delta_2 - \delta_1\delta_2 x_1 \\ 1-\delta_2 &= x_1(1-\delta_1\delta_2) \\ x_1 &= \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

Thus, the final allocation if agent 1 speaks first is:

$$\boxed{(x_1, 1-x_1) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \delta_2\frac{1-\delta_1}{1-\delta_1\delta_2}\right) = \left(\frac{3}{7}, \frac{4}{7}\right)} \tag{1}$$

(2) And if it is agent 2 who makes the first offer?

Agent 2's strategy is to offer x_2 and keep $1-x_2$ for himself. So, we continue from the above set of computations

$$x_2 = \delta_1 x_1 = \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2}$$

With the final allocation for 1 and 2 respectively being

$$\boxed{(x_2, 1-x_2) = \left(\delta_1\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right) = \left(\frac{2}{7}, \frac{5}{7}\right)} \tag{2}$$

(3) is there an agent that receives always more than the other? if yes, why? if no, why?

Agent 2's share is always greater than agent 1's, regardless of who speaks first. This occurs since $\delta_2 > \delta_2$, implying that agent 2 is more patient and discounts future consumption *less* than agent 1

(4) in general, how should relate the agents discount factors δ_1 and δ_2 for agent 1 to offer agent 2 a bigger share than his own?

This occurs if the following inequality exists between the allocations in equation (1)

$$\begin{aligned} 1 - x_1 &> x_1 \\ \delta_2 \frac{1-\delta_1}{1-\delta_1\delta_2} &> \frac{1-\delta_2}{1-\delta_1\delta_2} \\ 2\delta_2 - \delta_1\delta_2 &> 1 \end{aligned}$$

That is, when

$$\boxed{\delta_2 > \frac{1}{2-\delta_1}} \tag{3}$$

Q2. In Rubinstein's alternating offers bargaining game, is the unique subgame perfect equilibrium outcome continuous with respect to the agents' discount factors at $(\delta_1, \delta_2) = (1, 1)$?

Determining the continuity of the SPNE with $\delta_1 = \delta_2 = \delta = 1$ requires us to analyze the limits of the outcomes under 3 conditions

1. $\delta_1 = \delta_2 = \delta \to 1$

In this case, both agents are infinitely patient. If agent 1 speaks first

$$x_1 = \frac{1-\delta_2}{1-\delta_1\delta_2} = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta} \Rightarrow \lim_{\delta\to 1} \frac{1}{1+\delta} = \mathbf{\frac{1}{2}}$$

And if agent 2 speaks first

$$x_2 = \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2} = \frac{\delta}{1+\delta} \Rightarrow \lim_{\delta\to 1} \frac{\delta}{1+\delta} = \mathbf{\frac{1}{2}}$$

Thus, when two agents with the same discount rate become infinitely patient, they split the total in half thereby implying that $x_1 = 1 - x_1$ holds. Continuity requires that any sequence of outcomes converges to this solution.

2. $\delta_1 = 1, \delta_2 \to 1$

In order to ensure continuity, the limits of this sequence must lie on the same path as the above case.[2]

$$\begin{aligned} \mathbf{x_1} : \lim_{\delta_2\to 1} \frac{1-\delta_2}{1-\delta_1\delta_2} &= \lim_{\delta_2\to 1} \frac{-1}{-1} = 1 \\ \mathbf{1 - x_1} : \lim_{\delta_2\to 1} \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2} &= \lim_{\delta_2\to 1} \frac{-1}{-1} = 1 \end{aligned}$$

Obviously, $1 \neq 1 - 1$ and thus this particular limit does not lie on the same path as above. The same applies for the final case

[2]Given that the limits involve indeterminate forms, L'Hôpital's rule is implicitly applied in the computations below

3. $\delta_2 = 1, \delta_1 \to 1$

$$\begin{aligned} \mathbf{x_1} : \lim_{\delta_1 \to 1} \frac{1-\delta_2}{1-\delta_1\delta_2} &= \lim_{\delta_2 \to 1} \frac{0}{-1} = 0 \\ \mathbf{1-x_1} : \lim_{\delta_1 \to 1} \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2} &= \lim_{\delta_2 \to 1} \frac{0}{-1} = 0 \end{aligned}$$

Similarly, $0 \neq 1 - 1$ and thus, preferences defined by (δ_1, δ_2) are **not continuous** as the limits of the sequence do not converge to the same point.

Q3. Consider a single firm with production function f (with $f(0) = 0$, $f' > 0$, and $f'' < 0$) and a union deciding what part of time L to supply as labor l for the firm and what part $L - l$ to de vote to an outside option with reservation real wage w_0. Assume that $f'(L) > w_0$.

If the firm and the union are negotiating whether to implement the Nash bargaining solution allocation or the Kalai-Smorodinsky Bargaining solution allocation, what would be each party's most preferred choice?

Recall that a solution to a bargaining problem involves maximizing a particular *function* to obtain a point in, $u \in U$, the bargaining set. Before developing these functions, we must set up the bargaining set.

The utilities for both firms and workers (profits and income respectively) when they bargain over *wages*, **w** and *labour supply*, **l** and come to an agreement is

$$\begin{aligned} \text{Firm Profit: } \pi &= f(l) - wL \\ \text{Worker income} &= wl + w_0(L - l) \end{aligned}$$

In order to completely define the bargaining set, we also need to define the utilities when agents disagree i.e. cannot come to an agreement on (w, l)

$$\begin{aligned} \text{Firm Profit: } \pi &= f(0) - 0 = 0 \\ \text{Worker income} &= 0 + w_0(L - 0) = w_0 L \end{aligned}$$

Note that the firm gets 0 profit however, the workers still have their outside option paying at the reservation wage. Next, we assume that, regardless of the bargaining solution used, the (w, L) agreed on will be **efficient**. This is equivalent to the gradients $\nabla_i = \left(\frac{\partial}{\partial l}, \frac{\partial}{\partial w}\right)'$ being colinear. That is

$$\begin{bmatrix} \partial\pi/\partial l \\ \partial\pi/\partial w \end{bmatrix} = \lambda \begin{bmatrix} \partial R/\partial l \\ \partial R/\partial w \end{bmatrix} \Rightarrow \begin{bmatrix} f'(l) - w \\ -l \end{bmatrix} = -1 \begin{bmatrix} w - w_0 \\ l \end{bmatrix}$$

Where $\lambda = -1$ since the utilities of both agents move in opposite directions. Note that the bottom row (derivative wrt w. yields the trivial solution, $l = l$. However, the first row (derivative wrt to l) implies that for a bargaining solution to be efficient, we must have $f'(l^*) = w_0$ where whereby the marginal productivity of labour equals the marginal benefits of unemployment. Note that l^* is the optimal labour supply solution of the bargaining problem. However, given that we impose the additional condition that $f'(L) > w_0$ and knowing that $l \in [0, L]$ and that $f(\cdot)$ is strictly concave, arrive at a corner solution, stating that $f'(l) = w_0$ $\forall l \in [0, L]$. Obviously, the l maximizing the above is L giving us $f(L) = w_0 L$.

<u>Nash Bargaining Solution</u>

In order to compute the NBS for the wage and knowing that $l^* = L$ is the optimal labour supply, we construct the following Nash product[3]

$$w^N \in \arg\max_w \left[f(L) - w(L))\right] \cdot \left[wL - w_0 L\right] \tag{4}$$

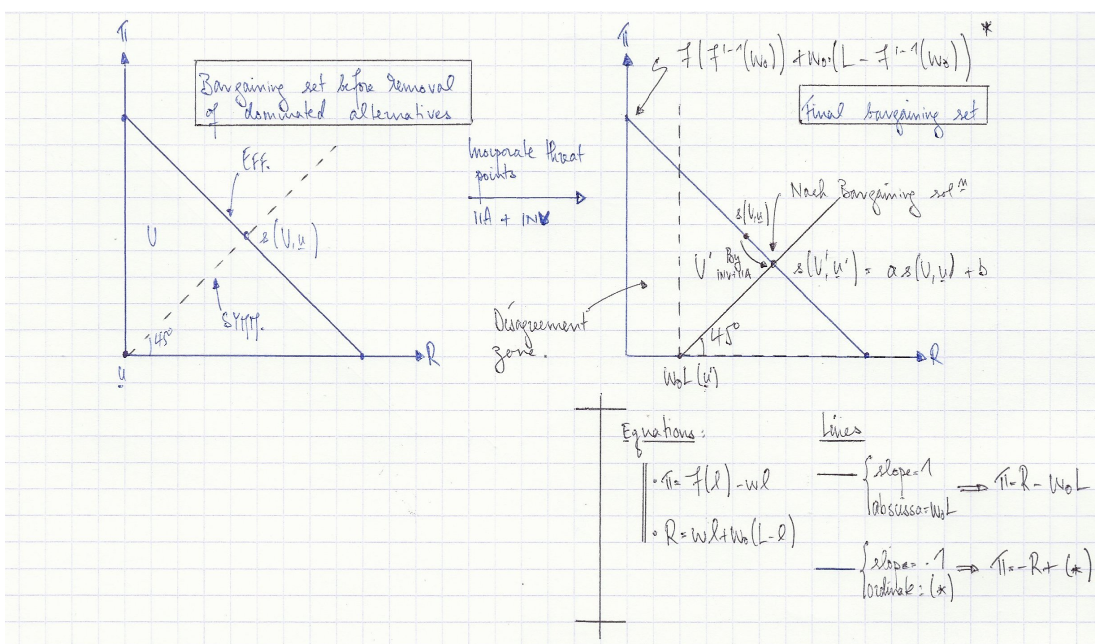
Which gives us the following first-order condition

$$\begin{aligned}
\partial w = 0 : \frac{\partial}{\partial w}\left[f(L)Lw - f(L)Lw_0 - L^2 w^2 + L^2 w_0 w\right] &= 0 \\
f(L)L - 2L^2 w + w_0 L^2 &= 0 \\
f(L) - 2Lw^N + w_0 L &= 0
\end{aligned}$$

Thus, the Nash bargaining solution is given by

$$\boxed{w^N = \frac{\frac{f(L)}{L} + w_0}{2}} \tag{5}$$

Extra: Note that this solution can also be obtained graphically via the axiomatic approach of Nash.



[3]Recall that the threat points have to be incorporated

The co-ordinate corresponding to the NBS can be computed via the intersection of the two lines

$$R - w_0 L = -R + f[f'^{-1}(w_0)] + w_0 \cdot (L - f'^{-1}(w_0)) \Rightarrow R = \frac{f[f'^{-1}(w_0)]}{2} - \frac{w_0 f'^{-1}(w_0)}{2} + w_0 L$$

Substituting in $R = wl + w_0(L - l)$ and using the fact that $l = f'^{-1}(w_0)$

$$\begin{aligned} w f'^{-1}(w_0) + \cancel{w_0 L} - w_0 f'^{-1}(w_0) &= \frac{f[f'^{-1}(w_0)]}{2} - \frac{w_0 f'^{-1}(w_0)}{2} + \cancel{w_0 L} \\ w^N f'^{-1}(w_0) &= \frac{f[f'^{-1}(w_0)] - w_0 f'^{-1}(w_0) + 2 w_0 f'^{-1}(w_0)}{2} \\ w^N &= \frac{f[f'^{-1}(w_0)]}{2 f'^{-1}(w_0)} + \frac{w_0 f'^{-1}(w_0)}{2 f'^{-1}(w_0)} \end{aligned}$$

Incorporating the corner solution $f'(L) = w_0 \Leftrightarrow L = f'^{-1}(w_0)$

$$\boxed{w^N = \frac{\frac{f(L)}{L} + w_0}{2}} \tag{6}$$

Kalai-Smorodinsky Solution

The KSsolution is characterized by equal proportional concessions of both parties from their respective maximally attainable utility levels. More precisely, for a given bargaining problem $(U, \underline{u})$ define the utopia point $u^\star = (u_F^\star, u_W^\star)$ by

$$u_i^\star = max\{u_i | u \in U, u_j \geq d_j \text{ for} j \neq i\} \quad i = F, W. \tag{7}$$

The first step, then is to find the highest attainable utility of each player. As before, we incorporate the corner solution $l = L$

$$\begin{aligned} \pi^{max} &= f(L) - w_0 L \\ R^{max} &= f(L) \end{aligned}$$

The first equation is obvious; at full employment, a firm would maximize it's profit setting the wage equal to the reservation wage (setting a lower wage however, would drive workers to the outside option). The second equation can be interpreted as; the highest wage a worker can demand such that the firm does not make a negative profits. That is, $\bar{w} | f(L) = wL \Rightarrow \bar{w} = \frac{f(L)}{L}$. The KS solution is then computed by setting the proportion of agents' utilities equal to the proportion of their maximum attainable utilities

$$\begin{aligned} \frac{\pi(w) - 0}{R(w) - w_0 L} &= \frac{\pi^{max} - 0}{R^{max} - w_0 L} \\ \frac{f(L) - wL}{wL - w_0 L} &= \frac{f(L) - w_0 L}{f(L) - w_0 L} \\ \frac{f(L) - wL}{wL - w_0 L} &= 1 \\ f(L) - wL &= wL - w_0 L \end{aligned}$$

Which, after solving for $w = w^{KS}$, gives us

$$\boxed{w^{KS} = \frac{\frac{f(L)}{L} + w_0}{2}} \tag{8}$$

Which is the same as the Nash Bargaining solution. Thus, firms and workers are indifferent between each bargaining solution as they obtain the same wage in each case.

If there was perfect competition, what allocation or allocations of profits would be equilibrium ones?

Taking wages as given, agents in perfect competition will individually choose the labour demand (for the firm) and labour supply (for the worker) that maximizes their respective utilities

<u>Firm</u>

$$\begin{aligned} \max_l \pi &= f(l) - w(l) \\ \text{FOC: } \partial_l &= f'(l) - w = 0 \\ \text{Value function: } \pi^* = f(l) - f'(l)l & \end{aligned}$$

Thus, the amount of labour demanded by the firm is $l^D = f'^{-1}(w)$.

<u>Workers</u>

$$\begin{aligned} \max_l R &= wl + w_0(L - l) \\ \text{FOC: } \partial_l &= w - w_0 = 0 \end{aligned}$$

Due to the corner solution given above, we define the labour supply as

$$l^S = \begin{cases} L \text{ if } w > w_0 \\ [0, L] \text{ if } w = w_0 \\ 0 \text{ if } w < w_0 \end{cases}$$

Knowing that $w > w_0$, we impose market clearing to compute the equilibrium wage

$$\begin{aligned} L &= f'^{-1}(w^\star) \\ w^* &= f'^{-1}(L) \end{aligned}$$

Q4 Prove that a bargaining solution s is independent of irrelevant strategies if, and only if, for any two bargaining problems $(\mathbf{U}, \underline{\mathbf{u}})$ and $(\mathbf{U}', \underline{\mathbf{u}}')$ such that

(1) $\underline{\mathbf{u}} = \underline{\mathbf{u}}'$ and
(2) $\mathbf{U} \subset \mathbf{U}'$

it holds that
(1') either $\mathbf{S}(\mathbf{U}', \underline{\mathbf{u}}') = \mathbf{S}(\mathbf{U}, \underline{\mathbf{u}})$
(2') or $\mathbf{S}(\mathbf{U}', \underline{\mathbf{u}}') \in \mathbf{U}' \cap \mathbf{U}^{\mathbf{C}}$.

Proof. Since this is an $\Leftrightarrow$ proof, we need to prove each direction

$\Rightarrow$: Suppose that a bargaining solution, $s(U, \underline{u})$ exists and satisfies properties (1) and (2) above[4]. In order to prove that this implies either (1') or (2'), we look at two cases for $S(U', \underline{u}')$

1. $s(U', \underline{u}') \in U$
This is shown graphically in the left figure below. Since by definition $s(U', \underline{u}') \in U'$, inclusion in U requires that there be one point where U intersects U'. This point defines the bargaining solutions for both sets and, given that this comprises one point, we must have that $\mathbf{S}(\mathbf{U}', \underline{\mathbf{u}}') = \mathbf{S}(\mathbf{U}, \underline{\mathbf{u}})$.

2. $s(U', \underline{u}') \notin U$

In this case, the two sets do not intersect at a point corresponding to the NBS. The right figure below shows that there exist a continuum of solutions under this condition. Since $s(U', \underline{u}') \notin U^C$ or U, the only feasible subset is defined by $U' \cap U^C$ and thus, $\mathbf{S}(\mathbf{U}', \underline{\mathbf{u}}') \in \mathbf{U}' \cap \mathbf{U}^{\mathbf{C}}$.

$\Leftarrow$
The inverse direction follows naturally from the previous argument. Depending on how the smaller set is positioned relative to the larger one ,we either have have a unique NBS for both sets (1') or an indeterminacy. Thus, IIA requires that we assume that $s(U', \underline{u}') \in U$ so as to ensure that the NBS does not change when removing irrelevant alternatives from the bargaining set.

□

[4]We make no assumptions regarding symmetry or efficiency

Q5 Prove that the bargaining solution

$$\mathbf{s(U,\bar{u})} = \arg\max_{\underline{\mathbf{u}}\leq \mathbf{u}\in \mathbf{U}} (\mathbf{u_1} - \underline{\mathbf{u}}_\mathbf{1})(\mathbf{u_2} - \underline{\mathbf{u}}_\mathbf{2}) \quad (9)$$

for all bargaining problem (U,$\underline{\mathbf{u}}$), is symmetric, efficient, and independent of irrelevant strategies.

Proof. The proof carried out in (Nash, 1950) shows that given the four *axioms* given above, there exists one bargaining solution that satisfies this which occurs as a result of maximizing the product of the players' gains in utility over the disagreement outcome.[5]

It is worthwile to note that since U is compact by definition and the objective function of (9) is continuous, there exists an optimal solution to (9). Moreover, given that the objective function is *strictly quasi-concave*, the optimal solution is unique. In order to prove that the Nash bargaining solution $s^N(U,\underline{u})$, is the unique bargaining solution that satisfies the four axioms, we first prove that the NBS satisfies the four axioms then show that if an arbitrary BS satisfies the four axioms, it must be equal to $s^N(U,\underline{u})$.

Step 1

1. **EFF**: This follows immediately from the fact that the objective function in (9) is increasing in u_1 and u_2.

2. **SYMM** Assuming that $\underline{u}_1 = \underline{u}_2$, let $u^\star = (u_1^\star, u_2^\star) = s^N(U,\underline{u})$. Then the permutation $(u_2^\star, u_1^\star)$ is also an optimal solution of (9). Given that this solution is unique by definition, it must be that $(u_1^\star = u_1^\star)$ and hence $s_1^N(U,\underline{u}) = s_2^N(U,\underline{u})$ thereby satisfying the symmetry property.

3. **IIA** Assuming a set $U' \subseteq U$. From the first two properties and given the definition of U' vis-'a-vis the superset U, it is apparent that $s_i^N(U,\underline{u}) \succeq_i s_i^N(U',\underline{u})\ \forall i$. As in Q4, if we impose that $s^N(U,\underline{u}) \in U'$. Thus, by definition of (9), $s^N(U,\underline{u})$ is optimal in U' and given the uniqueness of the solution, it must be that $s^N(U,\underline{u}) = s^N(U',\underline{u})$.

4. **INV** This axiom states that an affine transformation maintaining the same preference ordering should not alter the bargaining outcome. That is, suppose the bargaining problem defined by $(U,\underline{u})$ yields the NBS $s^N(U,\underline{u})$. Then, given the alternative BS $(U',\underline{u}')$ for some $\alpha > 0$, β:

$$\begin{aligned} U' &= \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) | (u_1,u_2) \in U\} \\ \underline{u}' &= (\alpha_1 \underline{u}_1 + \beta_1, \alpha_2 \underline{u}_2 + \beta_2) \end{aligned}$$

Then, $f_i(U',\underline{u}') = \alpha_i f_i(U,\underline{u}) + \beta_i\ \forall i$. In order to prove that the $s^N(U,\underline{u})$ satisfies INV, we apply the transformations directly to (9) to show that that $s^N(U,\underline{u}')$ is an optimal solution of the Nash product

[5] Note: The following exposition closely follows the proof given in the notes of the course 'Game Theory with Engineering Applications' at MIT (Course 6.254) - Lecture 14

$$
\begin{aligned}
(u_1 - \underline{u}_1)(u_2 - \underline{u}_2) = \alpha_1\alpha_2(u_1 - \underline{u}_1)(u_2 - \underline{u}_2) &= (\alpha_1 u_1 - \alpha_1\underline{u}_1)(\alpha_2 u_2 - \alpha_2\underline{u}_2) \\
&= (\alpha_1 u_1 + \beta_1 - \alpha_1\underline{u}_1 - \beta_1)(\alpha_2 u_2 + \beta_2 + \alpha_2\underline{u}_2 - \beta_2) \\
&= [\alpha_1 u_1 + \beta_1 - (\alpha_1\underline{u}_1 + \beta_1)] \cdot [\alpha_2 u_2 + \beta_2 - (\alpha_2\underline{u}_2 + \beta_2)] \\
&= [u'_1 - \underline{u}'_1] \cdot [u'_2 - \underline{u}'_2]
\end{aligned}
$$

Given the above computations, it is apparent that $s(U, \underline{u})$ maximizes the Nash product *if and only if* $s(U', \underline{u}')$ maximizes the NP over U'.

Step 2

Let $s(U, \underline{u})$ be an arbitrary bargaining solution satisfying the four axioms. This part of the proof requires us to show that $s(U, \underline{u}) = s^N(U, \underline{u})$.

To simplify, let $s^N(U, \underline{u}) = z$. Now, let us define a bargaining problem $(U', \underline{u}')$ that is obtained from $(U', \underline{u}')$ via the transformation (i.e., the specification of α and β) that map the threat point to the origin and the solution, $s^N(U, \underline{u})$ to the co-ordinate $(1/2, 1/2)$. Thus, we have

$$
\begin{aligned}
(\underline{u}'_1, \underline{u}'_2) &= (\alpha_1\underline{u}_1 + \beta_1, \alpha_2\underline{u}_2 + \beta_2) = (0, 0) \\
(u'_1, u'_2) &= (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) = (1/2, 1/2)
\end{aligned}
$$

Since s is INV by construction and s^N is INV from the 1st part of the proof, we have that

$$s(U, \underline{u}) = s^N(U, \underline{u}) \Leftrightarrow s(U', 0) = s^N(U', 0)$$

Note that we are trying to prove the LHS of the above equation however, given the *iff* relation, it suffices to show that $s(U', 0) = (u'_1, u'_2) = (1/2, 1/2)$.

Note that the point $(u'_1, u'_2) \in U'$ lies on the 2-simplex whereby $(u'_1 + (u'_2 = 1$, which, we claim defines the frontier of U'. That is, U' is bounded which implies that there does *not* exist a point (u_1, u_2) such that $u_1 + u_2) > 1$. Suppose, *a contrario* that $\exists \quad u \in U' : \quad u_1 + u_2) > 1$. We define the following convex combination between the two points in U' for $\lambda \in (0, 1)$

$$t = \lambda\,(1/2, 1/2) + (1 - \lambda)(u_1, u_2)$$

Since U' is, by definition, a convex set, we have that $t \in U'$. Notice that at the NBS, $s^N(U', 0) = (1/2, 1/2)$, $u_1 \cdot u_2 = 1/4$. Notice that as $\lambda \to 0$ in the equation above, the second term dominates the first thus, it is possible, for λ small enough, we can find $(t_1, t_2) \in U'$ such that $t_1 \cdot t_2 > 1$ thereby contradicting the optimality of t_1, t_2.

Having established the boundedness of U', we can define a rectangular set U'' s.t $U'' \supseteq U'$ that is symmetric w.r.t the line $u_1 = u_2$ and whose north-east frontier is the simplex defined above. As a result, the NBS point $s^N(U', 0) = (1/2, 1/2)$ is on the boundary of U''.

By 1. **EFF** and 2. **SYMM** we can 1. place $s(U'', 0)$ on the frontier of U'' and 2. place $s(U'', 0)$ on the 45°-line which gives the unique point, $s(U'', 0) = (1/2, 1/2)$. Finally, using **IIA**, given the properties already defined, we have that $s(U', 0) = s(U'', 0)\,(1/2, 1/2) \Rightarrow \;\; s(U', 0) = (1/2, 1/2)$. This concludes the proof.

□